International Journal of Theoretical Physics, Vol. 34, No. 3, 1995

Gel'fand–Dorfman Theorem and Exact Cocycle Poisson Structures

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Received April 28, 1994

We generalize the Gel'fand–Dorfman theorem to Poisson manifolds using the cohomological conditions. We find conditions to construct some compatible Poisson structures (exact cocycle type) suited to the needs of the theorem. Exact Poisson structures on vector spaces are also studied. We prove that every Lie–Poisson structure is exact.

1. INTRODUCTION

The purpose of this article is to find a technical tool to construct some functions in involution on the phase space of a mechanical system endowed with a pair of compatible Hamiltonian structures (or equivalently bi-Hamiltonian structure). Gel'fand and Dorfman (1979) proved an involution theorem on symplectic pairs on a manifold and later (Gel'fand and Dorfman, 1980) generalized it to Poisson pairs. Their generalization needs a certain factor space of differential one-forms to be trivial. In this article we generalize the Gel'fand–Dorfman theorem to Poisson manifolds using the Poisson cohomology spaces.

Later we discuss Poisson pairs satisfying the hypothesis of the above theorem. To construct such pairs, we consider a Poisson manifold and study exact 2-cocycles of the Poisson cohomology. Under certain conditions, they form Poisson pairs with the original structure and satisfy the hypothesis. We apply the above constructions to finite-dimensional vector space endowed with a Poisson structure. As an offshoot of this study we have observed that the exact 2-cocycle of the Poisson cohomology formed by the dilation vector field on a Lie–Poisson manifold (i.e., on the dual of a Lie algebra) gives

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0020-7748/95/0300-0443$07.50/0

rise to the Lie–Poisson structure itself. In other words, every Lie–Poisson structure is exact with the dilation vector field on the manifold.

2. GENERALIZED GEL'FAND–DORFMAN THEOREM

Let (M, P) be a Poisson manifold (Bhaskara and Viswanath, 1988b; Libermann and Marle, 1987) and let us denote the Poisson bracket of P by $\{\cdot, \cdot\}_P$. Let $P^\#: \Omega^1(M) \to \chi(M)$ be the map defined by $P^\#(\alpha)(\beta) = P(\alpha, \beta)$ for all $\alpha, \beta \in \Omega^1(M)$. [The notation, unless otherwise stated, is taken from Abraham and Marsden (1978).] We extend this map to any p-form on M and call it also $P^\#$, that is, for $\omega \in \Omega^p(M)$ we have

$$P^\#\omega(\alpha_1, \ldots, \alpha_p) = \omega(P^\#\alpha_1, \ldots, P^\#\alpha_p)$$

for $\alpha_i \in \Omega^1(M)$ and $P^\#(f) = f$ for all $f \in C^\infty(M)$.

There exists a Lie bracket on $\Omega^1(M)$ (Bhaskara and Viswanath, 1988a). We denote this bracket also by $\{\cdot, \cdot\}_P$. The Poisson cohomology operator on multivector fields on M is denoted by ∂_P (Bhaskara and Viswanath, 1988a). If Q is any other Poisson tensor, then $aP + bQ$ is again a Poisson tensor if and only if $[P, Q] = 0$, for $a, b \in \mathbf{R}$, where $[\cdot, \cdot]$ is the Schouten bracket (Bhaskara and Viswanath, 1988b). In such a case (P, Q) is called a Poisson pair (or pair of compatible Poisson structures) and the triplet (M, P, Q) is called a bi-Hamiltonian manifold.

Remark 1. (a) $[P^\#\alpha, P^\#\beta] = P^\#\{\alpha, \beta\}_P \; \forall \alpha, \beta \in \Omega^1(M)$ (Bhaskara and Viswanath, 1988a).

(b) $P^\# d\omega = -\partial_P P^\#\omega \; \forall \omega \in \Omega^p(M)$, for $p = 0, \ldots, n$ (Bhaskara and Viswanath, 1988b).

(c) $\{\cdot, \cdot\}_{P+Q} = \{\cdot, \cdot\}_P + \{\cdot, \cdot\}_Q$ (Bhaskara and Viswanath, 1988a).

Lemma. Suppose (M, P, Q) is a bi-Hamiltonian manifold and $\alpha, \beta, \gamma \in \Omega^1(M)$ satisfying $Q^\#(\beta) = P^\#(\alpha)$ and $Q^\#(\gamma) = P^\#(\beta)$. Then for any $\xi, \eta \in \Omega^1(M)$ we have

$$\begin{aligned} d\gamma(Q^\#(\xi), Q^\#(\eta)) - d\beta(Q^\#(\xi), P^\#(\eta)) - d\beta(P^\#(\xi), Q^\#(\eta)) \\ + d\alpha(P^\#(\xi), P^\#(\eta)) = 0 \end{aligned}$$

Proof. Using formula (6) in Table 2.4-1 on p. 121 of Abraham and Marsden (1978) and the hypothesis, we can write the left-hand side of the above equation as

$$\begin{aligned} -\gamma([Q^\#(\xi), Q^\#(\eta)]) + \beta([Q^\#(\xi), P^\#(\eta)]) + \beta([P^\#(\xi), Q^\#(\eta)]) \\ - \alpha([P^\#(\xi), P^\#(\eta)]) \end{aligned}$$

Note that

$$(P + Q)^{\#}(\{\xi, \eta\}_{P+Q})(\beta) = P^{\#}(\{\xi, \eta\}_P)(\beta) + Q^{\#}(\{\xi, \eta\}_Q)(\beta) + Q^{\#}(\{\xi, \eta\}_P)(\beta) + P^{\#}(\{\xi, \eta\}_Q)(\beta)$$

On the other hand, we have

$$\begin{aligned}(P + Q)^{\#}(\{\xi, \eta\}_{P+Q})(\beta) &= [P^{\#}(\xi) + Q^{\#}(\xi), P^{\#}(\eta) + Q^{\#}(\eta)](\beta) \\ &= [P^{\#}(\xi), P^{\#}(\eta)](\beta) + [P^{\#}(\xi), Q^{\#}(\eta)](\beta) \\ &\quad + [Q^{\#}(\xi), P^{\#}(\eta)](\beta) + [Q^{\#}(\xi), Q^{\#}(\eta)](\beta)\end{aligned}$$

Therefore we have that

$$[P^{\#}(\xi), P^{\#}(\eta)](\beta) + [P^{\#}(\xi), Q^{\#}(\eta)](\beta) - Q^{\#}(\{\xi, \eta\}_P)(\beta) - P^{\#}(\{\xi, \eta\}_Q)(\beta)$$

is equal to zero. This implies, by the hypothesis, that

$$-\gamma([Q^{\#}(\xi), Q^{\#}(\eta)]) + \beta([Q^{\#}(\xi), P^{\#}(\eta)]) + \beta([P^{\#}(\xi), Q^{\#}(\eta)]) - \alpha([P^{\#}(\xi), P^{\#}(\eta)])$$

is equal to zero.

Theorem. Let (M, P, Q) be a bi-Hamiltonian manifold such that the first ∂_Q-cohomology group is trivial and $\operatorname{Im} P^{\#} \subset \operatorname{Im} Q^{\#}$. Let $f_0, f_1 \in C^\infty(M)$ such that $Q^{\#}(df_1) = P^{\#}(df_0)$. Then:

(1) There exists a sequence of smooth functions $f_0, f_1, f_2, \ldots$ such that

$$Q^{\#}(df_{i+1}) = P^{\#}(df_i), \qquad i = 0, 1, 2, \ldots$$

(2) All f_i are in involution with respect to $\{\cdot, \cdot\}_P$ and $\{\cdot, \cdot\}_Q$.

Proof. (1) The proof is by induction. We construct the first step. A similar construction at every step gives the sequence. Let $\alpha = df_0$, $\beta = df_1$. Take a nonzero $\gamma \in Q^{\#-1}(P^{\#}(\beta))$. This is possible because $\operatorname{Im} P^{\#} \subset \operatorname{Im} Q^{\#}$. Using the lemma, we see that $d\gamma(Q^{\#}(\xi), Q^{\#}(\eta)) = 0\ \forall \xi, \eta \in \Omega^1(M)$. That is, $Q^{\#}\, d\gamma = 0 \Rightarrow -\partial_Q Q(\gamma) = 0$. Therefore there exists $g_2 \in C^\infty(M)$ such that $Q^{\#}(\gamma) = \partial_Q g_2$ since the first cohomology group is trivial. Therefore $Q^{\#}(\gamma) = \partial_Q g_2 = \partial_Q Q^{\#}(g_2) = -Q^{\#}(dg_2) = Q^{\#}(df_2)$, where $f_2 = -g_2$. A succession of this procedure leads to a sequence of functions which satisfies (1).

(2) For i, j $(i > j)$,

$$\begin{aligned}\{f_i, f_j\}_Q &= Q^{\#}(df_i)(df_j) = P^{\#}(df_{i-1})(df_j) = -P^{\#}(df_j)(df_{i-1}) \\ &= -Q^{\#}(df_{j+1})(df_{i-1}) = Q^{\#}(df_{i-1})(df_{j+1}) = \{f_{i-1}, f_{j+1}\}_Q\end{aligned}$$

Depending on whether $i - j$ is even or odd, we arrive either at $\{f_l, f_l\}_Q = 0$ or at $\{f_{l+1}, f_l\}_Q = Q^{\#}(df_{l+1})(df_l) = P^{\#}(df_l)(df_l) = 0$.

One can prove, in a similar way, that the f_i are in involution with respect to $\{\cdot, \cdot\}_P$.

The above lemma and the theorem are generalizations of Theorems 3.3 and 3.4 of Gel'fand and Dorfman (1979). Gel'fand and Dorfman proved them for symplectic structures and later generalized them to Poisson structures (the nomenclature is different there). In their generalization they need a certain factor space of differential 1-forms to be trivial. In our Poisson version we brought the Poisson cohomology into the picture, which sounds natural. One can see that our proof of the lemma, which is very crucial in constructing functions, is global geometric and the proof of the theorem goes along the same lines as theirs (Gel'fand and Dorfman, 1979, 1980). However, in the Poisson version one can find that it may be possible to have several sequences of functions in involution because at every step of induction there is a possibility to have several branchings by choosing each time a different nonzero element from $(Q^{\#-1}P^{\#})(\beta)$.

Examples. (1) Consider $M = so(3, \mathbf{R})^*$ and Q as the Lie–Poisson structure. Suppose X_1, X_2, X_3 is the basis such that

$$[X_1, X_2] = X_3, \qquad [X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2$$

Let x_1, x_2, x_3 be the coordinate system on M defined by the basis. Consider $R = X_1 \wedge X_2$, a unitary solution of the classical Yang–Baxter equation on $so(3, \mathbf{R})$. Then $ad(X_1) \wedge ad(X_2)$ defines a homogeneous quadratic Poisson structure on M (Bhaskara and Rama, 1991). Let us call this Poisson structure P. Since ad is a Lie algebra derivation P commutes with the Lie–Poisson structure Q (Bhaskara and Rama, 1991). A straightforward verification shows that P is determined by the following relations:

$$\{x_1, x_2\}_P = x_3^2, \qquad \{x_2, x_3\}_P = x_1x_3, \qquad \{x_3, x_1\}_P = x_2x_3$$

and

$$[P^{\#}] = x_3[Q^{\#}]$$

where $[\cdot]$ denotes the matrix of the tensor map inside. Since $x_3 = 0$ gives the set of all singularities of P, we get $\text{Im } P^{\#} = \text{Im } Q^{\#}$. Moreover, the first cohomology group of the Poisson manifold (M, Q) is trivial. Hence the theorem is applicable in this case.

(2) Consider $M = sl(2, \mathbf{R})^*$ and Q as the Lie–Poisson structure. Suppose H, X, Y is the basis such that

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H$$

Let h, x, y be the coordinate system on M defined by the basis. Consider R

$= H \wedge X$, a unitary solution of the classical Yang–Baxter equation on $sl(2, \mathbf{R})$. Then $ad(H) \wedge ad(X)$ defines a homogeneous quadratic Poisson structure on M (Bhaskara and Rama, 1991). Let us call this Poisson structure P. Since ad is a Lie algebra derivation P commutes with the Lie–Poisson structure Q (Bhaskara and Rama, 1991). A straightforward verification shows that P is determined by the following relations:

$$\{x, y\}_P = 2yh, \qquad \{y, h\}_P = -xh, \qquad \{h, x\}_P = 2h^2$$

and

$$[P^{\#}] = 2x[Q^{\#}]$$

where $[\cdot]$ denotes the matrix of the tensor map inside. Since $x = 0$ gives the set of all singularities of P, we get Im $P^{\#}$ = Im $Q^{\#}$. Moreover, the first cohomology group of the Poisson manifold (M, Q) is trivial. Hence the theorem is applicable in this case also.

(3) If we consider M to be any symplectic manifold whose first de-Rham cohomology group is trivial and Q to be any Poisson structure, then the theorem is applicable, for example, to an affine Poisson structure on an even-dimensional Lie algebra constructed with a nondegenerate 2-cocycle (Bhaskara, 1990).

The rest of the article is devoted to studying the hypothesis of the above theorem. The idea is to find some ways to produce bi-Hamiltonian manifolds with the required properties. The nature of the constructions being algebraic, we use Poisson algebras and multiderivations in the next section.

3. SOME COMPATIBLE POISSON STRUCTURES

Finding compatible Poisson structures is a usual practice. Here we give a construction suited to our needs. It is clear that if we have a Poisson structure Q and a vector field X such that $\partial_Q X$ is a Poisson structure (preferably different from Q), then they are naturally compatible. In this section we find some possibilities of $\partial_Q X$ becoming a Poisson structure. We also study the Poisson structures which are exact in their own cohomology. There are plenty of examples of both types.

Let $(\mathcal{A}, Q)$ be a Poisson algebra. That is, Q is an alternating 2-derivation on $\mathcal{A}$ such that $[Q, Q] = 0$ where $[\cdot, \cdot]$ is the Schouten bracket (Bhaskara and Viswanath, 1988b). Let us denote by ∂_Q the Lichnerowicz–Poisson cohomology operator (Lichnerowicz, 1977) on alternating multiderivations on $\mathcal{A}$. We write $Q(f, g) = \{f, g\}_Q$. Sometimes we also call $(\mathcal{A}, \{\cdot, \cdot\}_Q)$ a Poisson algebra. The typical example in mind is that $\mathcal{A} = C^\infty(M)$ for some Poisson manifold (M, Q).

Remark 2. For any $f, g \in \mathscr{A}$ and a 1-derivation X on $\mathscr{A}$, the following hold:

(a) $\partial_Q X(f, g) = \{f, X(g)\}_Q - \{g, X(f)\}_Q - X(\{f, g\}_Q)$.

(b) The Hamiltonian vector field X_f^Q of f is given by $X_f^Q(g) = \{g, f\}_Q$. In other words, $\partial_Q f = i_f Q = -X_f^Q$, where $i_f Q(g) = Q(f, g)$ for all g (Bhaskara and Viswanath, 1988b).

Q is called *exact* if $\partial_Q X = Q$ for some 1-derivation X on $\mathscr{A}$. An example of an exact Poisson structure is the following: consider $\mathscr{A} = C^\infty(\mathbf{R}^3)$ with coordinates x, y, z on $\mathbf{R}^3$. Let the Poisson structure Q be determined by the following relations:

$$\{x, y\}_Q = 1, \qquad \{y, z\}_Q = 0, \qquad \{z, x\}_Q = 0$$

For $X = x\, \partial/\partial x$ we have $\partial_Q X = Q$.

Sometimes $\partial_Q X$ can be a different Poisson structure. We call it *an exact cocycle Poisson structure*. A simple example of such a structure can be obtained by taking $X = \partial/\partial z$ with the above Poisson structure. Note that $\partial_Q X$ need not be a Poisson structure always. For example, consider $\mathscr{A} = C^\infty(\mathbf{R}^3)$ with coordinates x, y, z on $\mathbf{R}^3$. Let the Poisson structure Q be determined by the following relations:

$$\{x, y\}_Q = 2yz, \qquad \{y, z\}_Q = -xz, \qquad \{z, x\}_Q = 2z^2$$

It can be verified that for $X = \partial/\partial z$, $\partial_Q X$ is not a Poisson structure.

If P is another Poisson structure on $\mathscr{A}$, then we say both the structures are compatible if $[P, Q] = 0$. Since $\partial_Q^2 = 0$, if $\partial_Q X$ is a Poisson structure, then it is automatically compatible with P. In the following paragraphs we study the conditions for the existence of exact cocycle Poisson structures and some of their properties.

Proposition 1. Let $(\mathscr{A}, Q)$ be a Poisson algebra and let X be a 1-derivation on $\mathscr{A}$ such that $[X, [X, Q]] = 0$. Then $\partial_Q X$ is a Poisson structure.

Proof. Easy.

Example. Suppose $\mathscr{G}$ is a finite-dimensional Lie algebra and $\mathscr{G}^*$ is its dual. If we can identify $\mathscr{G}^{**}$ with $\mathscr{G}$, then the Lie–Poisson structure Q can be defined. That means $(C^\infty(\mathscr{G}^*), Q)$ is a Poisson algebra. Let $X_1, \ldots, X_n$ be a basis of $\mathscr{G}$ and let $x_1, \ldots, x_n$ be the coordinate system defined by $X_1, \ldots, X_n$ on $\mathscr{G}^*$. The Lie–Poisson structure is defined by

$$\{x_i, x_j\}_Q = \sum_{i,j,k} C_{ij}^k x_k$$

where C_{ij}^k are the structure constants of the Lie algebra with respect to the basis under consideration.

Now take any vector field A on $\mathcal{G}^*$ with constant coefficients. Then $[A, [A, Q]] = 0$. Indeed, if $A = \sum_l a_l\, \partial/\partial x_l$ for $a_l \in \mathbf{R}$, then

$$\begin{aligned}[A, [A, Q]](x_i, x_j)(\mu) &= L_A L_A Q(x_i, x_j)(\mu) \\ &= \frac{d^2}{dt\, ds}\bigg|_{t=s=0} Q(x_i, x_j)(\mu + t\alpha + s\alpha) \\ &= 0\end{aligned}$$

where α is treated as a vector in $\mathcal{G}^*$ with coordinates $a_1, \ldots, a_n$. Therefore, by Proposition 1, $\partial_Q A$ is a Poisson structure.

$f \in \mathcal{A}$ is called a *Casimir element* of Q if $\{f, g\}_Q = 0$ for all $g \in \mathcal{A}$. The space of all Casimir elements of $\mathcal{A}$ is denoted by $\mathcal{C}_Q$. By Remark 2(a), the following is true.

Proposition 2. Let $(\mathcal{A}, Q)$ be a Poisson algebra. Suppose $P = \partial_Q X$ is another Poisson structure on $\mathcal{A}$. Then:

1. $\{f, g\}_P = 0$ for all $f, g \in \mathcal{C}_Q$.
2. $X_f^P = X_{X(f)}^Q$ for all $f \in \mathcal{C}_Q$.

Proposition 3. Let $(\mathcal{A}, Q)$ be a Poisson algebra and let X be a 1-derivation on $\mathcal{A}$ such that $[X, X_f^Q] = 0$ for all $f \in \mathcal{A}$. Write $\partial_Q X = P$. Then P is a Poisson structure and $\operatorname{Im} P^\# \subset \operatorname{Im} Q^\#$.

Proof. Since $XX_f^Q = X_f^Q X$ for any f, once again by Remark 2(a), we get

$$\{f, X(g)\}_Q = -\{g, X(f)\}_Q = X(\{f, g\}_Q) = \{f, g\}_P$$

Now, it is easy to prove the Jacobi identity for $\{\cdot, \cdot\}_P$. We have $\operatorname{Im} P^\# \subset \operatorname{Im} Q^\#$ because $P^\#(df) = Q^\#(d(Xf))$ for all $f \in \mathcal{A}$.

Proposition 4. Let $(\mathcal{A}, Q)$ be a Poisson algebra. $Q = \partial_Q X$ for some 1-derivation X on $\mathcal{A}$ if and only if $[X, X_f^Q] = X_{X(f)}^Q - X_f^Q$ for all $f \in \mathcal{A}$.

Proof. Note that, by Remark 2(a), we have

$$\begin{aligned}\partial_Q X(f, g) &= -X_f^Q X(g) - X_{X(f)}^Q(g) + XX_f^Q(g) \\ &= [X, X_f^Q](g) - X_{X(f)}^Q(g)\end{aligned}$$

Now, if $\partial_Q X = Q$, then it is easy to see that $[X, X_f^Q] = X_{X(f)}^Q - X_f^Q$. On the other hand, by the generalized Jacobi identity (Bhaskara and Viswanath, 1988b), we have

$$[[Q, X], f] + [[X, f], Q] + [[f, Q], X] = 0$$

which implies that

$$i_f \partial_Q X = -X^Q_{X(f)} + [X, X^Q_f]$$

since $\partial_Q X = -[Q, X]$. When $[X, X^Q_f] = X^Q_{X(f)} - X^Q_f$, we have $i_f \partial_Q X = i_f Q$ for all f. Hence the proposition is true.

4. EXACT COCYCLE POISSON STRUCTURES ON VECTOR SPACES

Let V be a vector space with a coordinate system $x_1, \ldots, x_n$. Suppose

$$Q = \sum_{i,j} Q_{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$$

is a Poisson structure on V.

(A) We now study the hypothesis of Proposition 3 in this case. Assume that $X = \Sigma_l F_l\, \partial/\partial x_l$, where $F_l \in C^\infty(V)$.

For any $f \in C^\infty(V)$ the Hamiltonian vector field of f is given by

$$X^Q_f = -\sum_{i,j} Q_{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}$$

Note that

$$X^Q_{x_i} = -\sum_j Q_{ij} \frac{\partial}{\partial x_j}$$

Therefore

$$X^Q_f = \sum_i \frac{\partial f}{\partial x_i} X^Q_{x_i}$$

If $[X, X^Q_f] = 0$, we have

$$\sum_{i,l} F_l \frac{\partial^2 f}{\partial x_i \partial x_l} Q_{ij} = 0$$

for all j, from which one concludes that

$$F_i Q_{jk} + F_j Q_{ik} = 0$$

for all i, j, k. Therefore if $X = \Sigma_l F_l\, \partial/\partial x_l$ satisfies the above equation, then $\partial_Q X$ ($= P$ say) is a Poisson structure and $\operatorname{Im} P^\# \subset \operatorname{Im} Q^\#$.

(B) The next step is to find the consequences of Proposition 4. The necessary and sufficient condition for the exactness of a Poisson structure in terms of the coordinates is the following:

$$\sum_l \frac{\partial f}{\partial x_l} [X, X^Q_{x_l}] = \sum_{l,t} \frac{\partial F_t}{\partial x_l} \frac{\partial f}{\partial x_t} X^Q_{x_l} - \sum_{l,t} \frac{\partial f}{\partial x_l}$$

where

$$[X, X^Q_{x_l}] = \sum_j \left(F_j \frac{\partial Q_{lt}}{\partial x_j} - Q_{lj} \frac{\partial F_t}{\partial x_j} \right)$$

for all $t = 1, \ldots, n$.

Using these expressions, we can conclude that the necessary and sufficient condition in Proposition 4 is true if and only if

$$[X, X^Q_{x_l}] - \sum_t \frac{\partial F_t}{\partial x_l} X^Q_{x_t} + X^Q_{x_l} = 0$$

for all i.

(C) Now, let us assume that

$$Q_{ij} = \sum_{ijk} C^k_{ij} x_k$$

for $C^k_{ij} \in \mathbf{R}$. When $F_l = x_l$ it is easy to verify that $[X, X^Q_{x_l}] = 0$ and the necessary and sufficient condition for exactness is satisfied. Hence we have the following:

Proposition 5. Every Lie–Poisson structure is exact.

Finally, a word about the vanishing of the first Poisson cohomology group: One of the conditions in the hypothesis of the theorem is that the first cohomology group of (M, Q) should be trivial. In the following we give the information available on the vanishing of the Poisson cohomology. As far as we know, it is proved that the first Poisson cohomology spaces of the following Poisson manifolds are trivial:

(1) A symplectic manifold with its first de-Rham cohomology group is trivial (Bhaskara and Viswanath, 1988b; Lichnerowicz, 1977).

(2) The dual of a semisimple Lie algebra with its Lie–Poisson structure (Ginzburg and Weinstein, 1992).

(3) The dual group of a compact semisimple Poisson–Lie group (Ginzburg and Weinstein, 1992).

ACKNOWLEDGMENTS

K.H.B. and K.R. thank CONACyT and SNI for partial financial support and Ernesto Lacomba for encouragement and partial financial support from his research project.

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